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EXACT SOLUTION OF THE ANTIPLANE CONTACT PROBLEM FOR FINITE CANONICAL DOMAINS*

E.V. KOVALENKO, D.G. TARASOV and M.I. CHEBAKOV

The exact solution of a mixed problem of elasticity theory concerning pure shear by a stamp (in general, deformable) of a cylindrical body that occupies a domain bounded in section by coordinate lines of an orthogonal curvilinear coordinate system in a plane whose Lamé coefficients satisfy certain conditions, is obtained by constructing the closed solutions of integral equations of the first kind that contain Jacobi elliptic functions as kernels. Analogous problems were studied in /1, 2/, etc., in the special case of a strip and a ring. A scheme is proposed /1/ for constructing the exact solution of these problems by conformal mapping of the strip into a finite domain.

Many contact problems for bodies of finite size and periodic mixed problems /3, 4/ can be reduced to the integral equations under consideration. It is mentioned in /3/ that the integral operators of the equations obtained here can be inverted exactly by solving certain Riemann boundary-value problems for automorphic functions /5/.

1. Formulation of the problem. Consider a cylindrical body described in the coordinates α, β, z (α and β are curvilinear orthogonal coordinates in a plane) by the relationships $|\alpha| \leqslant R$, $B_1 \leqslant \beta \leqslant B_2$, $-\infty < z < \infty$. Let a stamp be clamped rigidly at the face $\beta = B_2$ in the domain $|\alpha| \leqslant A < R$ and move along the positive direction of the z-axis by under the action of a force T applied to each unit of its length; the face $\beta = B_1$ is clamped, while the faces $|\alpha| = R$ are clamped (problem A) or stress-free (problem B). Mathematically, the problem reduces to integration of the Lamé equation in coordinates α, β for the displacement w along the z axis

$$\Delta_{\alpha,\beta} w = 0, \quad \Delta_{\alpha,\beta} = \frac{1}{H_\alpha H_\beta}\left[\frac{\partial}{\partial\alpha}\left(h_{\beta\alpha}\frac{\partial}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(h_{\alpha\beta}\frac{\partial}{\partial\beta}\right)\right], \quad h_{\alpha\beta} = \frac{H_\alpha}{H_\beta} \tag{1.1}$$

where $H_\alpha = H_\alpha(\alpha, \beta)$, $H_\beta = H_\beta(\alpha, \beta)$ are Lamé coefficients of the curvilinear coordinates with the boundary conditions

$$w = 0 \quad (\beta = B_1) \tag{1.2}$$
$$w = 0 \quad (|\alpha| = R) \quad \text{(problem A)}$$
$$\tau_{\alpha z} = H_\alpha^{-1}\partial w/\partial\alpha = 0 \quad (|\alpha| = R) \quad \text{(problem B)}$$
$$w = D(\alpha) \quad (\beta = B_2, |\alpha| \leqslant A)$$
$$\tau_{\beta z} = H_\beta^{-1}\partial w/\partial\beta \quad (\beta = B_2, A < |\alpha| \leqslant R)$$

Here $D(\alpha)$ is the given displacement of the stamp as a function of α.

Following the method of separation of variables and setting $w(\alpha, \beta) = X(\alpha) \cdot Y(\beta)$, we obtain

$$h_{\beta\alpha}X''/X + X'/X\partial h_{\beta\alpha}/\partial\alpha = -[h_{\alpha\beta}Y''/Y + Y'/Y\partial h_{\alpha\beta}/\partial\beta]$$

Hence it follows that the variables in (1.1) are separated if one of the conditions

$$\partial h_{\beta\alpha}/\partial\alpha = 0 \tag{1.3}$$
$$\partial h_{\alpha\beta}/\partial\beta = 0 \tag{1.4}$$

is satisfied.

It is then best to replace (α, β, z) by (ξ, η, z), which follows for condition (1.3) from the satisfaction of the relation

$$\partial/\partial\eta = ah_{\alpha\beta}\partial/\partial\beta \tag{1.5}$$

and for condition (1.4), from the equality

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$$\partial/\partial\xi = ah_{\beta\alpha}\partial/\partial\alpha \tag{1.6}$$

Taking account of the relations (1.3) and (1.5), we find

$$\xi(\alpha) = \frac{\alpha}{a}, \quad \eta(\beta) = \frac{1}{a}\int_{B_1}^{\beta} h_{\beta\alpha}(\alpha, \beta)\, d\beta \tag{1.7}$$

$$a = A, \quad \vartheta = \frac{R}{a}, \quad \gamma = \frac{1}{a}\int_{B_1}^{B_2} h_{\beta\alpha}(\alpha, \beta)\, d\beta$$

and taking account of relations (1.4) and (1.6) we can write

$$\xi(\alpha) = \frac{1}{a}\int_0^{\alpha} h_{\alpha\beta}(\alpha, \beta)\, d\alpha, \quad \eta(\beta) = \frac{1}{a}(\beta - B_1) \tag{1.8}$$

$$a = \int_0^A h_{\alpha\beta}(\alpha, \beta)\, d\alpha, \quad \vartheta = \frac{1}{a}\int_0^R h_{\alpha\beta}(\alpha, \beta)\, d\alpha, \quad \gamma = \frac{1}{a}(B_2 - B_1)$$

As a result of these substitutions, the boundary-value problem (1.1) and (1.2) is reduced to a boundary-value problem in the coordinates ξ, η, z for the new function $u(\xi, \eta) = w(\alpha, \beta)$ $(\delta(\xi) = D(\alpha))$:

$$\Delta u = 0, \quad \Delta = \partial^2/\partial\xi^2 + \partial^2/\partial\eta^2 \tag{1.9}$$

$$u(\xi, \eta) = 0 \quad (\eta = 0)$$

$$u(\xi, \eta) = 0 \quad (|\xi| = \vartheta) \quad \text{(problem A)}$$

$$\partial u(\xi, \eta)/\partial\xi = 0 \quad (|\xi| = \vartheta) \quad \text{(problem B)}$$

$$u(\xi, \eta) = \delta(\xi) \quad (\eta = \gamma, \ |\xi| \leqslant 1)$$

$$\partial u(\xi, \eta)/\partial\eta = 0 \quad (\eta = \gamma, \ 1 \leqslant |\xi| \leqslant \vartheta)$$

2. ***Construction of solutions of the boundary-value problems (1.9).*** Using the Fourier series representation of the solutions we reduce problems A and B to finding a function $\varphi(\xi)$ from the integral equation

$$\int_0^1 [l(x+\xi) + l(x-\xi)]\,\varphi(x)\, dx = \delta(\xi) \quad (0 \leqslant \xi \leqslant 1) \tag{2.1}$$

which is connected in the cases (1.3) and (1.5) with the contact stresses under the stamp by the relation

$$\varphi(\xi) = aH_{\alpha}G^{-1}\tau_{\beta z}(\alpha, B_2) \tag{2.2}$$

and in the cases (1.4) and (1.6) by the expression

$$\varphi(\xi) = aH_{\beta}G^{-1}\tau_{\beta z}(\alpha, B_2) \tag{2.3}$$

The kernel $l(y)$ can be represented in the form

$$l(y) = C_0 + \frac{1}{\vartheta}\sum_{k=1}^{\infty} \frac{\operatorname{tg}\lambda_k\gamma}{\lambda_k}\cos\lambda_k y \tag{2.4}$$

$$C_0 = 0, \quad \lambda_k = \frac{2k-1}{2}\frac{\pi}{\vartheta} \ \text{(problem A)}, \quad C_0 = \frac{\gamma}{2\vartheta}, \quad \lambda_k = \frac{k\pi}{\vartheta} \ \text{(problem B)}$$

The series (2.4) can be summed by using expansions of the Jacobi elliptic functions in the parameter $q = \exp[-\pi K'(k)/K(k)]$ /6/ where k is the modulus of the elliptic functions, $K(k)$ is the complete elliptic integral of the second kind, and $K'(k) = K(\sqrt{1-k^2})$.

We will have $(\varkappa = K(k)\,\vartheta^{-1})$

$$l(y) = \frac{1}{2\pi}\ln\frac{1+\operatorname{cn}\varkappa y}{1-\operatorname{cn}\varkappa y} \quad \text{(problem A)} \tag{2.5}$$

$$l(y) = \frac{1}{2\pi}\ln\frac{1+\operatorname{dn}\varkappa y}{1-\operatorname{dn}\varkappa y} \quad \text{(problem B)}$$

Here $\operatorname{sn} y$, $\operatorname{cn} y$, $\operatorname{dn} y$ are Jacobi elliptic functions of modulus k, determined from the transcendental equation

$$K'(k)/K(k) = \gamma/\vartheta$$

Later problem A will be examined, while the expressions for problem B can be obtained by replacing $\operatorname{cn} y$ and $\operatorname{dn} y$ by $\operatorname{dn} y$ and $\operatorname{cn} y$, respectively.

Differentiating (2.1) with respect to ξ we obtain $(l'(y) = k(y))$

$$\int_0^1 [k(x+\xi) + k(x-\xi)]\,\varphi(x)\,dx = \delta'(\xi) \quad (0 \leqslant \xi \leqslant 1) \tag{2.6}$$

$$k(y) = \operatorname{dn} \varkappa y / \operatorname{sn} \varkappa y$$

Using the evenness of the function $\varphi(\xi)$ and the relationship /7/ between the Jacobi functions, we convert the integral Eqs.(2.6) to an equation with a Cauchy kernel by using the change of variables $\varepsilon = \operatorname{sn} \varkappa\xi / \operatorname{sn} \varkappa$, $\zeta = \operatorname{sn} \varkappa x / \operatorname{sn} \varkappa$

$$\int_{-1}^{1} g(\zeta) \frac{d\zeta}{\zeta - \varepsilon} = \pi \psi(\varepsilon) \quad (|\varepsilon| \leqslant 1) \tag{2.7}$$

$$g(\zeta) = \varphi(x)/\operatorname{dn} \varkappa x, \quad \psi(\varepsilon) = \delta'(\xi)/\operatorname{dn} \varkappa\xi \tag{2.8}$$

whose solution is known /8/ and has the form

$$g(\varepsilon) = \frac{1}{\pi\sqrt{1-\varepsilon^2}} \left(N_0 - \int_{-1}^{1} \frac{\psi(\zeta)\sqrt{1-\zeta^2}}{\zeta - \varepsilon} d\zeta\right) \tag{2.9}$$

Taking (2.8) into account, we obtain

$$\varphi(\xi) = \frac{\operatorname{dn} \varkappa\xi \operatorname{sn} \varkappa}{\pi\theta(\xi)} \left[N_0 - \frac{1}{\varkappa \operatorname{sn} \varkappa} \int_{-1}^{1} \frac{\delta'(\tau) \operatorname{cn} \varkappa\tau\, \theta(\tau)}{\operatorname{sn} \varkappa\tau - \operatorname{sn} \varkappa\xi} d\tau\right] \tag{2.10}$$

$$\theta(y) = \sqrt{\operatorname{sn}^2 \varkappa - \operatorname{sn}^2 \varkappa y}$$

The quantity N_0 is determined by substituting the solution (2.10) into (2.1).

3. ***Absolutely rigid stamp.*** If it is assumed that $\delta(\alpha) = \delta = \text{const}$, then (2.10) is simplified and the value of the constant N_0 can be written in the explicit form

$$N_0 = \frac{\pi\varkappa\delta}{K(\operatorname{cn} \varkappa) \operatorname{sn} \varkappa}, \quad \varphi(\xi) = N_0 \frac{\operatorname{dn} \varkappa\xi \operatorname{sn} \varkappa}{\pi\theta(\xi)} \tag{3.1}$$

As $|\xi| \to 1$ the solution (3.1) found has a root singularity with the coefficient

$$\omega = \lim_{|\xi| \to 1} \varphi(\xi) \sqrt{1-\xi^2} = N_0 \frac{\operatorname{sn} \varkappa \operatorname{dn} \varkappa}{\pi\varkappa}$$

When the side boundary $\alpha = R$ tends to infinity, $k \to 1$, the elliptic functions change into hyperbolic functions $\operatorname{sn} y \to \operatorname{th} y$; $\operatorname{cn} y$, $\operatorname{dn} y \to \operatorname{sech} y$, but $\varkappa \to \pi(2\gamma)^{-1}$. We hence obtain the well-known solution of the contract problem of the shear of a "layer" /9/.

The contact stresses under the stamp $\tau_{\beta z}$ are determined by means of (2.2) or (2.3), taking the change of variables (1.7) or (1.8), respectively, into account. The connection between the shearing force T acting on the stamp and its displacement δ is here given by the expression

$$T = \int_{-A}^{A} \tau_{\beta z}(\alpha, B_2)\, d\alpha = G \int_{-1}^{1} \varphi(\xi) H_\alpha^{-1}(\alpha(\xi), B_2)\, d\xi \tag{3.2}$$

where $\alpha(\xi) = a\xi$ in conformity with (1.7), while the function $\alpha(\xi)$ in the case (1.8) is found as the solution of the differential equation $d\alpha/d\xi = ah_{\beta\alpha}(\alpha, \beta)$.

In the cases of rectangular, as well as bipolar, elliptic, parabolic, and hyperbolic coordinates, the Lame coefficients are identical and, therefore the changes of variables (1.7) and (1.8) are linear.

In particular, for the rectangular coordinates $(\alpha = x, \beta = y)$

$$H_\alpha = H_\beta = 1, \quad a\tau_{\beta z}(\alpha, B_2) = G\varphi\left(\frac{\alpha}{a}\right), T = GN_0 \frac{\operatorname{sn} \varkappa}{\pi\varkappa} K'(\operatorname{cn} \varkappa)$$

For bipolar coordinates (q is half the distance between the poles)

$$H_\alpha = H_\beta = \frac{q}{\operatorname{ch}\alpha + \cos\beta} = H, \quad \tau_{\beta z}(\alpha, B_2) = \frac{G}{aH}\varphi\left(\frac{\alpha}{a}\right)$$

while the quantity T is determined from (3.2).

For polar coordinates $(\alpha = \varphi,\ \beta = r)$ we can write

$$H_\alpha = \beta, \quad H_\beta = 1, \quad \frac{\partial}{\partial\alpha}(h_{\beta\alpha}) = 0, \quad \xi(\alpha) = \frac{\alpha}{a}, \eta(\beta) = \frac{1}{a}\ln\frac{\beta}{B_1}, \quad a = A,$$

$$\vartheta = \frac{R}{A}, \quad \gamma = \frac{1}{a}\ln\frac{B_2}{B_1}, \quad \tau_{\beta z}(\alpha, B_2) = \frac{G}{a\beta}\varphi\left(\frac{\alpha}{a}\right)$$

The force T has the form (3.2).

Remarks. Let us find the connection between the constants N_0 in (3.1) and

$$P = \int_{-1}^{1} \varphi(x)\,dx \tag{3.3}$$

Substituting the second formula of (3.1) into (3.3), we obtain

$$P = 2\pi^{-1}N_0\varkappa^{-1}\operatorname{sn}\varkappa\, K'(\operatorname{cn}\varkappa) \tag{3.4}$$

When the right-hand side of the original integral Eq.(2.1) differs from a constant, it is best to use Theorem 3.7 in /9/ to determine the constant N_0 in the solution (2.10). Taking account of the relationships (3.1), (3.2) and (3.4), we will have

$$P = \frac{\varkappa}{K(\operatorname{cn}\varkappa)}\int_{-1}^{1}\frac{\delta(\tau)\operatorname{dn}\varkappa\tau}{\theta(\tau)}\,d\tau \tag{3.5}$$

Using the method presented in Sect.2, the odd modification of the integral Eq.(2.1) can be investigated (with $l(x \pm \xi)$ replaced by $l(x \mp \xi)$). Without giving the details, we write

$$\varphi(\xi) = -\frac{\varkappa\operatorname{cn}\varkappa\xi}{\pi\theta(\xi)}\int_{-1}^{1}\frac{\delta'(\tau)\operatorname{dn}\varkappa\tau\theta(\tau)}{\operatorname{sn}\varkappa\tau - \operatorname{sn}\varkappa\xi}\,d\tau$$

This result has no physical meaning but often turns out to be useful in combination with (2.10), (3.1)-(3.5) for regularizing integral equations of the first kind of two-dimensional contact problems for domains bounded by the coordinate lines of certain systems of curvilinear coordinates in a plane /4/.

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